A Quasi-Linear Parabolic Partial Differential Equation with Accretive Property

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ABSTRACT

A quasi-linear parabolic, partial differential equation is investigated in a Banach space by converting such equation into an abstract Cauchy problem. Analysis is, thereby carried out using the fundamental results according to Browder in the theory of accretive operators. This shows that the problem is $M$-accretive and establishes that this partial differential equation has a solution.

KEYWORDS: Cauchy Problem, Banach space, Accretive Operator

INTRODUCTION

Existence theorems fall into two classes; constructive and non-constructive. People wishing to use mathematics to prove existence theorems are probably right to the non-constructive proofs. They correctly assume think that proving that something exists, without providing a way of establishing its existence, will not help much.

Constructive existence proofs are quite a different matter. They almost always consist of two parts: an approximation procedure and an error estimate. In other words “constructive existence proofs give a recipe for finding an approximation to what you want, and a formula for estimating how good the approximation is” [16].

The existence is then proven by taking a sequence of better and better approximations, and showing that they converge using the error estimate. Historically, the approximation procedure was mainly a theoretical tool.

On the existence and uniqueness of solution to differential equations, Lipschitz’ work provides a more general condition for existence and uniqueness than the requirement that \( \frac{\partial f}{\partial x} \) be continuous, as quoted in most elementary texts. A differential equation \( x' = f(t, x) \) admits a Lipschitz condition if the function \( f \) admits a Lipschitz condition. The importance of this is that “the Lipschitz constant \( k \) bounds the rate at which solution can pull apart” [16]. As the following computation shows: if in the region \( A \), \( u_1(t) \) and \( u_2(t) \) are solutions to the differential equation \( x' = f(t, x) \) then they pull apart at a rate
\[ |u_1(t) - u_2(t)| = |f(t, u_1(t)) - f(t, u_2(t))| \leq k |u_1(t) - u_2(t)| \]

So in practice, we will want the smallest possible value of \( k \). As we will see, such a number \( k \) controls the numerical solutions of differential equation at the rate at which errors compound.

The Cauchy problem for some equations can be treated by different methods. One first proves existence of global weak solution by compactness method using the energy estimates. The general abstract methods for studying the Cauchy problem was developed by [21], which consists of recasting the Cauchy problem in the form of an integral equation, and solving that equation by a “fixed point method”. A less general method was proposed by [10]. In [13], a Cauchy problem for a quasi-linear hyperbolic first order linear partial differential equation was studied. Also [6], studied the longtime behaviour and regularity of solutions for stark equations and showed that for a class of short-range potentials, the gain of smoothness and the decay as time approaches infinity, are close to those of the corresponding Schrödinger equation. [3] considered the problem for a parabolic distributed control system with feedback control constraints, and proved the existence of a periodic trajectory. Also [14] considered the Cauchy problem for non-linear Schrödinger equation and non-linear Klein-Gordon equation and proved the existence of global weak solutions by compactness method using the energy estimates. [6] studied the longtime behaviour and regularity of solution for stark evolution equation and revealed that, as time approaches infinity, the gains of smoothness and the decay are close to those of the corresponding Schrödinger equation. In [13] a Cauchy problem for a quasi-linear hyperbolic, first order partial differential equation in Banach space, was considered and shown to be m-accretive. While [2] investigated a second order evolution equation with dynamic boundary conditions, and established a regularity result for an abstract Cauchy problem in its applicability to problems modeling the dynamic vibrations of linear viscoelastic rods and beams. [20] studied a method to calculate periodic solutions of functional differential equations, using a method to obtain explicitly periodic solutions of some types of functional differential equations. [8] considered a technique for proving existence of solutions, using a sequence of appropriate functions defined equally into sub-intervals and then proved that the solution of the limit function exists. [11] Also investigated an existence theorem for evolution inclusions involving opposite monotonicities, and proved the existence of a strong global solution. Furthermore [1] analysed a continuous dependence on modeling for related Cauchy problems of a class of evolution equation, and regularization problems that may be ill-posed, against errors made in formulating the governing equations of mathematical models. [7] worked on existence theorems for certain elliptic and parabolic semi-linear equations, and obtained existence theorems for both parabolic and elliptic equations with zero initial value given explicit conditions on non-homogeneous terms. [4] considered a periodic solution of functional differential equations of the neutral type, and gave necessary and sufficient conditions for the existence of periodic solutions for convex functional differential equations of neutral type with finite and infinite delay. In [23], the existence of solution for a class of generalized evolution equation on closed set was investigated using approximating solutions and proved existence theorems for solutions to an autonomous generalized evolution equation. [12] considered the problem for a necessary and sufficient condition for the solutions of a functional differential equation and showed that every solution of the non-homogeneous functional differential equation is non-decreasing, Liptchizian and satisfy a sub-linear condition which is either oscillatory or tends to zero asymptotically. Also [15] studied the oscillatory behaviour of nth-order forced functional differential equation and found that the solution exists on some ray and are non-trivial near infinity. Furthermore, [5] investigated the approximations of solutions to non-linear Sobolev type of evolution equations, and established the existence and uniqueness of solutions to every approximate integral equation.
using a fixed-point argument. [22] worked on the existence and stability of non-linear functional differential equations, and obtained the existence and stability results for a class of non-linear functional differential equations. Also [19] studied the well-posedness of evolution problems of the mechanics of Visco-Plastic media and obtained the existence, uniqueness and continuous dependence on the initial data of solutions of boundary-value problems arising in the mechanics of homogeneous visco-plastic media. Furthermore, [17] proved the persistence of invariant sets for dissipative evolution equations and showed that results concerning the persistence of invariant sets of ordinary differential equations under perturbation may be applied directly to a certain class of partial differential equations.

This work considers a quasi-linear parabolic partial differential equation, defined in a Banach space and show that it has a solution.

Some General Concepts

Let $X$ be a real Banach space with norm $\| \cdot \|$ and dual $X^*$. An operator with domain $D(A)$ and range $R(A)$, in $X$ is said to be accretive, if for all $x_1, x_2 \in D(A)$ and $r > 0$ holds the inequality [9].

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(Ax_1 - Ax_2)\| - - - (2.1.1)$$

Also from [18] $A$ is accretive if and only if for $x_1, x_2 \in D(A)$, there is $j \in J(x_1 - x_2)$ such that $\langle Ax_1 - Ax_2, j \rangle \geq 0$ where

$$J(x) = \{ x^* \in X^* \mid x^* \ll x, \|x^*\|^2 = \|x\|^2, \exists x^* \in X, \}$$

is the normalized duality mapping of $X$ and $\ll \cdots \|$ denotes the duality pairing between $X$ and $X^*$. If $X^*$ is uniformly convex, the duality mapping is uniformly continuous on bounded subsets of $X^*$. An accretive operator, $A$ is said to be $m$-accretive if $R(I + mA) = X$ for all $r > 0$, where $I$ is the identity operator on $X$. In terms of the concept of contractions, an operator $A$ is said to be accretive if $(I + rA)^{-1}$ is a contraction for $r \geq 0$, that is, if

$$\| (x_1 + rAx_1) - (x_2 + rAx_2) \| \geq \| x_1 - x_2 \| - - - (2.1.2)$$

If $x$ is a Hilbert space, the accretive condition (2.1.2) reduces to

$$\text{Re}\langle Ax_1 + Ax_2, x_1 - x_2 \rangle \geq 0$$

for all $x_1, x_2 \in X - - - - - - (2.1.3)$

Lemma [13]

Let $X$ be a real Banach space and let $A : D(A) \subset X \to X$ be an $m$-accretive operator such that $u = (A + n^{-1})x_1, v = (A + n^{-1})x_2$ where $u, v \in X; x_1, x_2 \in D(A); n \in R^+$. Then operator

$(A + n^{-1}) : X \to D(A)$ is continuous and bounded.

Proof

Given $u, v \in X; x_1, x_2 \in D(A); n \in R^+$ as defined above we have

$$\| (A + n^{-1})x_1 - (A + n^{-1})x_2, j \| = \frac{1}{n} \|x_1 - x_2\|^2$$

for $j \in J(x_1 - x_2)$.

This implies that

$$\| (A + n^{-1})^{-1}u - (A + n^{-1})^{-1}v \| \leq n\|u - v\|,$$

which shows the continuity of $(A + n^{-1})^{-1}

Now let

$v = 0, x_0 \in D(A)$ with $0 = (A + n^{-1})x_0$,

then we have

$$\| (A + n^{-1})^{-1}u \| \leq n(\|u\| + \|x_0\|).$$

This proves the boundedness of the operator $(A + n^{-1})^{-1}$.

Hyperbolic Operator Problem


$$\begin{align*}
u_t + (f(u))_x &= 0, \quad 0 < x < 1 \\
u(0, x) &= u_0(x), \quad 0 < x < 1 \\
u(t, 0) &= 0, \quad t > 0.
\end{align*}$$

and established that it admits a solution, assuming $f$ is Lipschitz continuous.

Using these general concepts together with the work done in the hyperbolic case, some results for the parabolic partial differential equation can be established.
Existence of Solution of quasi-linear parabolic equation
Consider a quasi-linear parabolic partial differential equation

\[ u_t - (f(u))_{xx} = 0, \quad 0 < x < 1 \]
\[ u(0, x) = u_0(x), \quad 0 < x < 1 \]
\[ u(t, 0) = u(t, 1) = 0, \quad t > 0. \]

The above problem would now be transformed into the differential operator as below, from which analysis can be carried out

**Theorem**

Let \( X = L [0,1], \) let \( f : R \rightarrow R \) be continuous, strictly increasing and \( f(0) = 0 \) and let

\[ D(A) = \{ u \in C[0,1], u(0) = u(1) = 0, f(u) \text{ and } f'(u) \}, \]

are absolutely continuous. \( Au = -(f(u))'' \), then \( A \) is m – accretive.

**Proof**

\[ \int_0^1 (A(u) - A(v))g(f(u) - f(v))dx = \]
\[ \int_0^1 -(f(u) - f(v))''g(f(u) - f(v))dx \]

Using integration by parts, we have

\[ u = g(u - f(v)), \quad du = g'(u - f(v))(f(u) - f(v))dx \]

\[ dv = -(f(u) - f(v))''dx, \quad v = -(f(u) - f(v))' \]

\[ \frac{1}{2}g(f(u) - f(v))(f(u) - f(v))'\frac{1}{2} + \frac{1}{2}g(f(u) - f(v))'\frac{1}{2}g(f(u) - f(v))dx \]

\[ \frac{1}{6}g(f(u) - f(v))(f(u) - f(v))'\frac{1}{2}g(f(u) - f(v))dx = \frac{1}{6}g(f(u) - f(v))(f(u) - f(v))'\frac{1}{2}g(f(u) - f(v))dx \]

we observe that

\[ (u - v)g(f(u) - f(v)) = |u - v|g(f(u) - f(v)) \]

And on the assumption of \( g \) and equation 3.2 it follows that

\[ \int_0^1 (u - v) + r(Au + Av)dx \]

\[ \geq \int_0^1 |u - v|g(f(u) - f(v)) + r(Au - Av)g(f(u) - f(v))dx \]

\[ \geq \int_0^1 |u - v|g(f(u) - f(v)) + r(Au - Av)g(f(u) - f(v))dx \]

\[ \geq \int_0^1 |u - v|g(f(u) - f(v))dx \]

If \( g = g_n \) in the equation (3.4) where

\[ g_n(s) \] signs, \( |s| \geq \frac{1}{n} \]

And let

\[ n \rightarrow \infty, (u - v)g_n(f(u) - f(v)) \rightarrow |u - v|, \]

then from equation 3.4 follows that

\[ \|(u - v) + r(Au - Av)\| \geq \|u - v\| \]

Which shows that \( A \) is accretive.

Now it remains to show that it is m-accretive, that is \( R(I + rA) = L[0,1] \) for all \( r > 0 \), where \( I \) is the identity operator, \( R \) is range. Without loss of generality \( r \) may be assumed to be unity. As in the lemma in [13] on a Cauchy problem for a quasi-linear hyperbolic partial differential equation, the operator \( A \) is continuous and bounded.

Hence given \( h \in L [0,1] \), consider

\[ u \in D(A) \text{ such that } u + Au = h. \]

Now let

\[ v = f(u) \text{ and } \beta(v) = f^{-1}. \]

so that

\[ \beta(v) = v' - h, \quad v(0) = v(1) = 0 \]

Since \( A \) is accretive then if \( \beta(v) \in D(A) \) and \( \beta(v) = v' - h \), then,

\[ |\beta(v)|_{L [0,1]} \leq \| \beta(v) - v ' - h(0) - 0 \|_{L [0,1]} = \| h \|_{L [0,1]} \]

Also by the mean value theorem, for some \( \xi \in (0,1), \) \( v' \xi = 0 \) since

\[ v(0) = v(1) = 0 \]

\[ \left| v'(x) \right| \leq \int_0^1 v''(s)ds \]

\[ \leq \int_0^1 |v''(s)|ds \leq \int_0^1 v''(s)ds \leq 2\| h \|_{L [0,1]} \]

Since the operator \( A \) has been shown to be accretive, and since

\[ R(I + rA) = L[0,1] \quad \text{for} \ r > 0 \]
then is m-accretive. Hence by [10] the operator $A$ for the quasi-linear parabolic problem admits a solution, that is, there exists a solution for this parabolic partial differential equation problem.

**DISCUSSION OF THE RESULTS**

From [9] a fundamental result in the theory of accretive operator, an initial value problem (ivp) is solvable in a Banach space $X$ if the operator problem describing the (ivp) is locally Lipschitzian and accretive on $X$.

Here the Banach space $X = L^2[0,1]$ with the domain of the operator $A$ as $C^2[0,1]$ a dense subset of $X$, under the conditions as given in theorem 3.1 in respect of $f$, our operator $A$ is not only accretive but also m-accretive.

In the analysis, it has been established that the Cauchy problem of this quasi-linear parabolic partial differential equation is accretive and infact m-accretive, assuming the Lipschitzian condition is satisfied.

**CONCLUSION**

The m-accretiveness of the quasi-linear parabolic partial differential equation problem has been established, just as it was observed in the hyperbolic case as in [13].

This quasi-linear parabolic partial differential equation, under certain conditions of the operator has a solution using the fundamental result of [9].

**RECOMMENDATIONS**

Although this problem can be solved directly, the method of transforming a partial differential equation problem into an operator problem can still be improved; some assumptions could be further investigated, such as applying the Lipschitz condition to a similar problem to show that the operator is m-accretive.

**REFERENCES**


